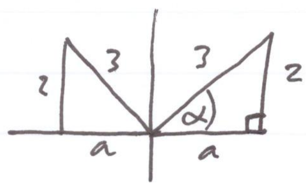
Mathematics : The core course for A-level

Bostock & Chandler

Chapter 7 : Trigonometric Identities

Miscellaneous Ex 7

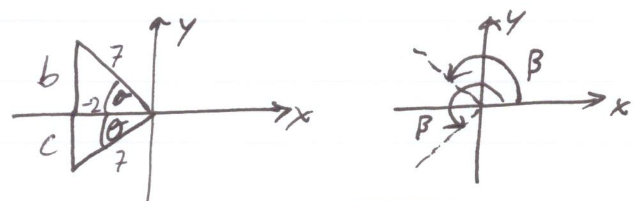
① a) $\sin\alpha = \frac{2}{3}$, geometrically this can be seen as



By Pythagoras' Theorem : $a = \sqrt{9-4} = \sqrt{5}$

So $\cos\alpha = \frac{\sqrt{5}}{3}$ or $\cos\alpha = -\frac{\sqrt{5}}{3}$

Similarly if $\cos\beta = -\frac{2}{7}$. Since cos is negative, β lies in quadrant II or III :



So by Pythagoras' Theorem, $b = c = \sqrt{49-4} = \sqrt{45}$

$\therefore \sin\theta = \frac{\sqrt{45}}{7} \Rightarrow \sin\beta = +\frac{\sqrt{45}}{7}$; $\sin\beta = -\frac{\sqrt{45}}{7}$

So
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \frac{\sqrt{5}}{3}\cdot\left(-\frac{2}{7}\right) - \left(\frac{2}{3}\right)\cdot\left(\frac{\sqrt{45}}{7}\right)$$
$$= -8\sqrt{5}/21$$

or

$$\cos(\alpha+\beta) = \frac{\sqrt{5}}{3}\cdot\left(-\frac{2}{7}\right) - \left(\frac{2}{3}\right)\cdot\left(-\frac{\sqrt{45}}{7}\right)$$

Similarly $\cos(\alpha+\beta) = \left(-\frac{\sqrt{5}}{3}\right)\cdot\left(-\frac{2}{7}\right) - \left(\frac{2}{3}\right)\left(\frac{\sqrt{45}}{7}\right)$

$= -4\sqrt{5}/21$

or

$$\cos(\alpha+\beta) = \left(-\frac{\sqrt{5}}{3}\right)\cdot\left(-\frac{2}{7}\right) - \left(\frac{2}{3}\right)\cdot\left(-\frac{\sqrt{45}}{7}\right)$$

$$= 8\sqrt{5}/21$$

b) given $3\cos\theta - 5\sin\theta = 2$

let $3\cos\theta - 5\sin\theta = R\cos(\theta+\alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\therefore \left.\begin{array}{l} 3 = R\cos\alpha \\ 5 = R\sin\alpha \end{array}\right\} \Rightarrow R = \sqrt{25+9} = \sqrt{34}$

and $\tan\alpha = \frac{5}{3} \Rightarrow \alpha \simeq 59.036^\circ$

So $\sqrt{34}\cos(\theta + 59.036^\circ) = 2$

$$\cos(\theta + 59.036^\circ) = \frac{2}{\sqrt{34}}$$

$$\theta + 59.036^\circ = \pm 69.94^\circ \pm 360^\circ n \quad , n = 0, 1, 2, \ldots$$

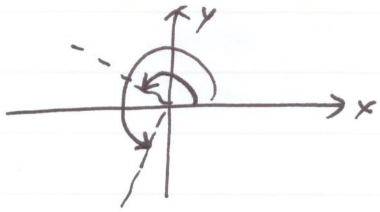
$\therefore \theta = 10.904^\circ \pm 360^\circ n$

or $\theta = -128.98^\circ \pm 360^\circ n$

Since $\theta \in [-180^\circ; 180^\circ]$, $\theta \simeq 10.9^\circ, -129^\circ$

(2) (a) i) given $\cos\frac{3x}{4} = \tan 163^\circ = -0.305730681$

Here cos is negative, implying $\frac{3x}{4}$ lies in quadrant II or III



So $\quad \frac{3x}{4} = \cos^{-1}(-0.305730681) = 107.8021288,$ in quad II

or $\quad \frac{3x}{4} = 252.1978712$ in quad III

$\therefore\ x = 336.26^\circ$ or 143.74°

ii) given $7\cos x - 24\sin x = 12.5$

Let $\quad 7\cos x - 24\sin x = R\cos(x+\alpha)$

$\qquad = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore\ \left.\begin{array}{l} R\cos\alpha = 7 \\ R\sin\alpha = 24 \end{array}\right\} \Rightarrow R = \sqrt{7^2+24^2} = 25$

$\alpha = \tan^{-1}\frac{24}{7} = 73.74^\circ$

So $\quad 25\cos(x + 73.74^\circ) = 12.5$

$\Rightarrow \quad \cos(x + 73.74^\circ) = \frac{1}{2}$

$\Rightarrow \quad x + 73.74^\circ = \pm 60^\circ \pm 360^\circ n, \quad n = 0, 1, 2, \ldots$

So $\quad x = (60 - 73.74) + 360 = 346.26^\circ$

& $\quad x = (-60 - 73.74) + 360 = 226.26^\circ$

(b) given $\sin^2 \frac{\pi}{8} - \cos^4 \frac{3\pi}{8}$ we have That

$$\frac{4\pi}{8} - \frac{\pi}{8} = \frac{3\pi}{8}$$

$$\cos \frac{3\pi}{8} = \cos\left(\frac{4\pi}{8} - \frac{\pi}{8}\right) = \sin \frac{\pi}{8}$$

So $\sin^2 \frac{\pi}{8} - \cos^4 \frac{\pi}{8} = \sin^2 \frac{\pi}{8} - \sin^4 \frac{\pi}{8}$

$$= \sin^2 \frac{\pi}{8}\left(1 - \sin^2 \frac{\pi}{8}\right)$$

$$= \sin^2 \frac{\pi}{8} \cdot \cos^2 \frac{\pi}{8}$$

$$= \left(\sin \frac{\pi}{8} \cdot \cos \frac{\pi}{8}\right)^2$$

$$= \left(\frac{1}{2} \sin 2 \cdot \frac{\pi}{8}\right)^2 = \frac{1}{4}\left(\sin \frac{\pi}{4}\right)^2$$

$$= \frac{1}{4} \cdot \frac{2}{4} = \frac{1}{8}$$

(3) given $\sin A = \frac{3}{5}$ in quadrant I, and $\sin B = \frac{5}{13}$

(a)

in quadrant II, Then

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$= \cos A \cdot \cos B - \left(\frac{3}{5}\right) \cdot \left(\frac{5}{13}\right)$$

By Pythagoras we end up with

quadrant I: $\cos A = \frac{\sqrt{5^2 - 3^2}}{5} = \frac{4}{5}$

quadrant II: $\cos B = -\frac{\sqrt{13^2 - 5^2}}{13} = -\frac{12}{13}$

So

$$\cos(A+B) = \frac{4}{5} \cdot \left(-\frac{12}{13}\right) - \left(\frac{3}{5}\right) \cdot \left(\frac{5}{13}\right) = -\frac{63}{65}$$

And $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B} = \dfrac{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}{1 + \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}}$

$$= \frac{\frac{3/5}{4/5} - \frac{5/13}{-12/13}}{1 + \frac{3/5}{4/5} \cdot \frac{5/13}{(-12/13)}} = \frac{3/4 + 5/12}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{7}{11}$$

(b) For $\tan\theta + 3\cot\theta = 5\sec\theta$

So $\frac{\sin\theta}{\cos\theta} + 3\frac{\cos\theta}{\sin\theta} = 5 \cdot \frac{1}{\cos\theta}$

$\Rightarrow \quad \sin^2\theta - 5\sin\theta + 3(1-\sin^2\theta) = 0,$

by $\cos^2\theta + \sin^2\theta = 1$.

Hence $\quad 2\sin^2\theta + 5\sin\theta - 3 = 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta + 3) = 0$

$\therefore \quad 2\sin\theta - 1 = 0 \quad \& \quad \sin\theta = -3$

But $\sin\theta = -3$ is not valid, hence

$2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2}$

$\therefore \theta = 30^\circ$ and 150°

$= \frac{\pi}{6}$ and $\frac{5\pi}{6}$ in $0 < \theta < 2\pi$

(4) (a) $\quad 2\sin\theta + \cos\theta = 1$

let $\quad 2\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$\therefore \quad 2 = R\cos\alpha \quad \& \quad 1 = R\sin\alpha$

So $\quad R = \sqrt{4+1} = \sqrt{5}$ and $\tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26.57^\circ$

So $\quad 2\sin\theta + \cos\theta = \sqrt{5}\sin(\theta + 26.57^\circ) = 1$

$\therefore \quad \sin(\theta + 26.57^\circ) = \frac{1}{\sqrt{5}} \Rightarrow \theta + 26.57^\circ = 26.57^\circ, 153.43^\circ$

$\therefore \theta = 0, 126.86^\circ$ in $[0, 180^\circ]$

(b) $2\sin\theta + \cos 2\theta = 1$

$\Rightarrow 2\sin\theta + 1 - 2\sin^2\theta = 1$

$\therefore 2\sin^2\theta - 2\sin\theta = 0.$

$\therefore \sin\theta(\sin\theta - 1) = 0$

$\Rightarrow \sin\theta = 0$ or / and $\sin\theta = 1$

Hence $\theta = 0^\circ, 180^\circ$ or/and $\theta = 90^\circ$

(5) (a) given $\tan^2\theta = 5 - \sec\theta$ we have

$\sec^2\theta - 1 = 5 - \sec\theta$ (by $1 + \tan^2\theta = \sec^2\theta$)

$\therefore \sec^2\theta + \sec\theta - 6 = 0$

$\Rightarrow (\sec\theta + 3)(\sec\theta - 2) = 0$

$\therefore \sec\theta = -3$ or $\sec\theta = +2$

$\Rightarrow \cos\theta = -\frac{1}{3}$ or $\cos\theta = \frac{1}{2}$

hence $\theta = \pm 109.47^\circ$ or $\theta = \pm 60^\circ$ for $0 \le \theta \le 180^\circ$

(b) given $6\cos\theta + 7\sin\theta = 4$,

let $6\cos\theta + 7\sin\theta = R\sin(\theta + \alpha)$

$$= R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$$

$\therefore\ 6 = R\sin\alpha$ and $7 = R\cos\alpha$

$\Rightarrow 36 + 49 = R^2\sin^2\alpha + R^2\cos^2\alpha$

So $\sqrt{85} = R^2 \Rightarrow R = +9.22$

Also $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{6}{7} \Rightarrow \alpha = 40.6^\circ$

So $\sqrt{85}\,.\sin(\theta + 40.6^\circ) = 4$

$\therefore\ \sin(\theta + 40.6^\circ) = 0.434$

So $\theta + 40.6^\circ = 25.71^\circ \pm 360n$

& $\theta + 40.6^\circ = 154.29^\circ \pm 360n$; $n = 0, 1, 2, \cdots$

So $\theta = (25.71 - 40.6) + 360 = 345.11^\circ$ in $0 < \theta < 36$

& $\theta = 154.29 - 40.6 = 113.69^\circ$ in $0 < \theta < 360$

⑥ ⓐ given $\sin 2x + 2\cos 2x = 1$

Let $\sin 2x + 2\cos 2x = R.\sin(2x+\alpha)$

$= R\sin 2x \cos\alpha + R\sin\alpha \cos 2x$

$\therefore\ 1 = R\cos\alpha$ and $2 = R\sin\alpha$

So $1^2 + 2^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$\Rightarrow\ 5 = R^2,\ \therefore R = \sqrt{5}$

Also $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = 2 \Rightarrow \alpha = 63.43^\circ$

So $\sqrt{5}.\sin(2x + 63.43^\circ) = 1$

$\therefore\ \sin(2x + 63.43^\circ) = \frac{1}{\sqrt{5}}$

$\Rightarrow\ 2x + 63.43^\circ = 26.57^\circ \pm 360^\circ n$

and $2x + 63.43^\circ = 153.43^\circ \pm 360^\circ n$

$\therefore\ 2x = -36.86^\circ \pm 360^\circ n = 143.14^\circ \pm 360^\circ n$

& $2x = 90^\circ \pm 360^\circ n$

$\Rightarrow\ x = 161.57^\circ \pm 180^\circ n$

& $x = 45^\circ \pm 180^\circ n$

So for $x \in [0^\circ, 360^\circ]$, $x = 161.57^\circ, 45^\circ; 225^\circ, 341.57^\circ$

(b) given $6\cos\theta + 7\sin\theta = 4$,

let $6\cos\theta + 7\sin\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$\therefore\ 6 = R\sin\alpha$ and $7 = R\cos\alpha$

$\Rightarrow\ 36 + 49 = R^2\sin^2\alpha + R^2\cos^2\alpha$

So $\sqrt{85} = R^2 \Rightarrow R = +9.22$

Also $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{6}{7} \Rightarrow \alpha = 40.6^\circ$

So $\sqrt{85}\cdot\sin(\theta + 40.6^\circ) = 4$

$\therefore\ \sin(\theta + 40.6^\circ) = 0.434$

So $\theta + 40.6^\circ = 25.71^\circ \pm 360n$

& $\theta + 40.6^\circ = 154.29^\circ \pm 360n$; $n = 0, 1, 2, \ldots$

So $\theta = (25.71 - 40.6) + 360 = 345.11^\circ$ in $0 < \theta < 36$

& $\theta = 154.29 - 40.6 = 113.69^\circ$ in $0 < \theta < 360$

(b) given $\cos 3x + \cos x = \sin 2x$

So $4\cos^3 x - 3\cos x + \cos x = 2\sin x \cos x$

So $4\cos^3 x - 3\cos x + \cos x - 2\sin x \cos x = 0$

$\therefore \quad 2\cos x(2\cos^2 x - 1 - \sin x) = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad 2\cos^2 x - 1 - \sin x = 0$

$\therefore \quad 2(1-\sin^2 x) - 1 - \sin x = 0$

$\therefore \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin x = \frac{1}{2}$ or $\sin x = -1$

So $x = \pm 360^\circ n$ or $x = 30^\circ \pm 360^\circ n$ or $x = 270^\circ \pm 360^\circ n$
& $x = 150^\circ \pm 360^\circ n$

(7) a) given $\sin(\theta - \alpha) = k\sin(\theta + \alpha)$ then

$\sin\theta\cos\alpha - \sin\alpha \cdot \cos\theta = k\sin\theta\cos\alpha + k\sin\alpha\cos\theta$

$\therefore \quad \sin\theta\cos\alpha\,(1-k) = \sin\alpha . \cos\theta\,(1+k)$

$\therefore \quad \frac{\sin\theta}{\cos\theta}(1-k) = \frac{\sin\alpha}{\cos\alpha}(1+k) \Rightarrow \tan\theta = \frac{(1+k)}{1-k} . \tan\alpha$

When $h = \frac{1}{2}$ & $\alpha = 150^\circ$ we have

$$\tan\theta = 3\tan 150^\circ = -3\frac{\sqrt{3}}{3} = -\sqrt{3}$$

$$\therefore \theta = -60^\circ \pm 360^\circ n$$

For $0^\circ \le \theta \le 360^\circ$ we have $\theta = 120^\circ, 300^\circ$

(b) given $\cos 3x = \sin 2x$ & $x = \frac{\pi}{10}$ we have

$$4\cos^3 x - 3\cos x = 2\sin x\cos x$$

$$\therefore 4\cos^3 x - 3\cos x - 2\sin x\cos x = 0$$

$$\therefore \cos x\left(4\cos^2 x - 3 - 2\sin x\right) = 0$$

$$\therefore \cos x\left(4(1-\sin^2 x) - 3 - 2\sin x\right) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad 4 - 4\sin^2 x - 3 - 2\sin x = 0$$

$$\Rightarrow 4\sin^2 x + 2\sin x - 1 = 0 \qquad (*)$$

Which I assume is there equation $4S^2 + 2S - 1 = 0$.
But they haven't defined S so I assume $S = \sin x$!

$$\therefore \sin x = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

But $\frac{\pi}{10}$ is in quadrant I which makes $\sin$ positive so

$$\sin x = \frac{-1+\sqrt{5}}{4} \Rightarrow x = \sin^{-1}\left(\frac{-1+\sqrt{5}}{4}\right) = \frac{\pi}{10}$$

(8) (a) given $\sin 2\theta = \sin\frac{\pi}{6}$

Then $\sin 2\theta = \frac{1}{2}$

$\therefore \quad 2\theta = \frac{\pi}{6} \pm 2n\pi$

& $\quad 2\theta = \frac{5\pi}{6} \pm 2n\pi$

So $\quad \theta = \frac{\pi}{12} \pm n\pi$ & $\theta = \frac{5\pi}{12} \pm n\pi$

for $0 \le \theta \le 2\pi$, $\theta = \frac{\pi}{12}$ & $\frac{13\pi}{12}$, and $\frac{5\pi}{12}$ & $\frac{17\pi}{12}$

(b) given $(2\cos\theta + 3\sin\theta)^2 \le 13$

Then $\quad 2\cos\theta + 3\sin\theta \le \pm\sqrt{13}$

But $\quad 2\cos\theta + 3\sin\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$\therefore \quad 2 = R\sin\alpha$ and $3 = R\cos\alpha$

From which $R^2\sin^2\alpha + R^2\cos^2\alpha = 4 + 9 = 13$

$\therefore R^2 = 13 \Rightarrow R = \pm\sqrt{13}$

So $\quad \pm\sqrt{13}\cdot\sin(\theta + \alpha) \le \pm\sqrt{13}$ is true for all θ

(9) given $\sin\theta + \sin 3\theta = \cos\theta + \cos 3\theta$

use the factor formulae :

$\therefore \quad 2\sin 2\theta \cos\theta = 2\cos 2\theta \cos\theta$

$\therefore \quad 2\cos\theta\,(\sin 2\theta - \cos 2\theta) = 0$

$\Rightarrow \quad \cos\theta = 0 \quad$ or $\quad \sin 2\theta - \cos 2\theta = 0$

i.e $\quad \tan 2\theta = 1$

$\therefore \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad$ and $\quad 2\theta = \frac{\pi}{4} \pm n\frac{\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{8} \pm n\frac{\pi}{4}$

hence $\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

(10) given $\quad 4\cos\theta - 3\sin\theta \equiv 5\cos(\theta + \alpha)$

Then we have $\quad 4\cos\theta - 3\sin\theta = 5\cos\theta\cos\alpha - 5\sin\theta\sin\alpha$

$\therefore \quad 4 = 5\cos\alpha \quad$ and $\quad 3 = 5\sin\alpha$

$\Rightarrow \quad \tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.87^\circ = 36^\circ 52'$

For $f(\theta) \equiv 4\cos\theta - 3\sin\theta - 4 = 5\cos(\theta + 36^\circ 52') - 4 \;;\; \theta \in [-180, 180]$

i) max value of $\cos x$ is $+1$ at $x = 0$: $\theta + 36^\circ 52' = 0$, $\therefore \theta = -36^\circ 52'$

ii) min " " " -1 at $x = 180^\circ$ & -180° : $\theta + 36^\circ 52' = \pm 180^\circ$

So $\theta = 143^\circ 7'$ or $-216^\circ 52'$, but $\theta \in [-180^\circ, 180]$ so $\theta = 143^\circ 7'$

iii) Zero value of $\cos x$ is 0 at $x = 90^\circ$ & -90° See Next Page $\longrightarrow$

$f(\theta) = 4\cos\theta - 3\sin\theta - 4 = 0$

$\Rightarrow \quad 4\cos\theta - 3\sin\theta = 4$

$\Rightarrow \quad 5\cos(\theta + 36^\circ 52') = 4$

$\Rightarrow \quad \cos(\theta + 36^\circ 52') = \frac{4}{5}$

$\Rightarrow \quad \theta + 36^\circ 52' = \pm 36^\circ 52'$ in $[-180^\circ, 180^\circ]$

$= 0^\circ$ or $-73^\circ 44'$

(11) a) $(\sin 2\theta - \sin\theta)(1 + 2\cos\theta) = \sin 2\theta + 2\sin 2\theta\cos\theta$
$- \sin\theta - 2\sin\theta\cos\theta$

$= 2\sin 2\theta\cos\theta - \sin\theta$

$= 2 \times 2\sin\theta\cos^2\theta - \sin\theta$

$= 4\sin\theta(1 - \sin^2\theta) - \sin\theta$

$= 4\sin\theta - 4\sin^3\theta - \sin\theta$

$= 3\sin\theta - 4\sin^3\theta$ ✓

$= \sin 3\theta$

(b) given $3\cos x + 1 = 2\sin x$ we have

$3\cos x - 2\sin x = -1$

let $3\cos x - 2\sin x = R\cos(x + \alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore \quad 3 = R\cos\alpha \quad$ and $\quad 2 = R\sin\alpha$

$\therefore \quad R = \sqrt{9+4} = \sqrt{13} \quad$ and $\tan\alpha = \frac{2}{3}$

$\Rightarrow \alpha = 33.69^\circ$

So we have $\quad \sqrt{13}\cos(x + 33.69^\circ) = -1$

$\therefore \quad (x + 33.69^\circ) = \pm 106.1^\circ \pm 360^\circ n, \; n = 1, 2, ..$

So $\quad x = 106.1 - 33.69^\circ = 72.41^\circ$

But also $\quad x + 33.69^\circ = \pm 253.897^\circ \pm 360^\circ n, \; n = 1, 2, 3.$

So $\quad x = 253.897 - 33.69^\circ = 220.21^\circ$

(12) (a) $\quad 3\sin\theta + 4\cos\theta = 2$

Let $\quad 3\sin\theta + 4\cos\theta = R\sin(\theta + \alpha)$

$= R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$\therefore \quad 3 = R\cos\alpha \quad$ and $\quad 4 = R\sin\alpha$

$\Rightarrow R = \sqrt{9+16} = 5 \quad \nearrow \quad \tan^{-1}\alpha = \frac{4}{3} \Rightarrow \alpha = 53.13^\circ$

So $\quad 5\sin(\theta + 53.13^\circ) = 2$

$\therefore \quad \sin(\theta + 53.13^\circ) = \frac{2}{5} \Rightarrow \quad \theta + 53.13^\circ = 23.58^\circ \pm 360^\circ n$

and $\theta + 53.13^\circ = 156.42^\circ \pm 360^\circ n$

where $n = 1, 2, \ldots$

(13) To Prove $\sec x + \tan x = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

RHS : $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{\tan \pi/4 + \tan x/2}{1 - \tan\frac{\pi}{4} \cdot \tan\frac{x}{2}}$

$= \frac{1 + \tan x/2}{1 - \tan x/2}$ (*)

LHS : $\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{\cos x + \cos x \cdot \sin x}{\cos^2 x}$

$= \frac{\cos x\,(1 - \sin x)}{1 - \sin^2 x} = \frac{\cos x\,(1 - \sin x)}{(1 + \sin x)(1 - \sin x)}$

$= \frac{\cos x}{1 - \sin x}$

$= \frac{\cos(x/2 + x/2)}{1 - \sin\left(\frac{x}{2} + \frac{x}{2}\right)}$

$= \frac{\cos^2 x/2 - \sin^2 x/2}{1 - 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}}$

$= \frac{1 - \tan^2\frac{x}{2}}{\sec^2\frac{x}{2} - 2\tan\frac{x}{2}}$

$= \frac{1 - \tan^2 x/2}{\left(1 + \tan^2\frac{x}{2}\right) - 2\tan\frac{x}{2}} = \frac{1 - \tan^2 x/2}{(1 - \tan x/2)^2}$

$= \frac{(1 + \tan x/2)(1 - \tan x/2)}{(1 - \tan x/2)^2}$

$= \frac{1 + \tan x/2}{1 - \tan\frac{x}{2}} = RHS$ (*)

So $\theta = 23.58 - 53.13 = -29.55^\circ$

& $\theta = 156.42 - 53.13 = 103.29^\circ$

(b) $7\tan 2\theta + 4\sin\theta = 0 \Rightarrow 7\dfrac{\sin 2\theta}{\cos 2\theta} + 4\sin\theta = 0$

$\Rightarrow 7\sin 2\theta + 4\sin\theta\cos 2\theta = 0$

$\Rightarrow 7 \times 2\sin\theta\cos\theta + 4\sin\theta(\cos^2\theta - \sin^2\theta) = 0$

$\Rightarrow 14\sin\theta\cos\theta + 4\sin\theta(1 - 2\sin^2\theta) = 0$

$\Rightarrow 2\sin\theta(7\cos\theta + 2(1 - 2\sin^2\theta)) = 0$

$\Rightarrow 2\sin\theta(7\cos\theta + 2 - 4(1 - \cos^2\theta)) = 0$

$\Rightarrow 2\sin\theta(7\cos\theta + 2 - 4 + 4\cos^2\theta) = 0$

$\Rightarrow 2\sin\theta(4\cos^2\theta + 7\cos\theta - 2) = 0$ (*)

For the quadratic we have

$$\cos\theta = \frac{-7 \pm \sqrt{49 + 32}}{8} = -\frac{7}{8} \pm \frac{9}{8} = -2, \frac{1}{4}$$

But $\cos\theta \neq -2$, $\therefore \cos\theta = \frac{1}{4} \Rightarrow \theta = \pm 75.52 \pm 360^\circ n$

So $\theta = \pm 75.52$ in $[-180, 180^\circ]$

and by (*) $\sin\theta = 0 \Rightarrow \theta = 0, 180^\circ$

To show a similar Expression for $\sec x - \tan x$ let x be $-x$ in $\sec x + \tan x$. Hence

$$\sec(-x) + \tan(-x) = \tan\left(\frac{\pi}{4} + \left(\frac{-x}{2}\right)\right)$$

$$\therefore \quad \sec x - \tan x = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

Now $\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{4} + \frac{2\pi/3}{2}\right)$

$\therefore$ do $\sec\frac{2\pi}{3} + \tan\frac{2\pi}{3} = -2 - \sqrt{3} = -(2+\sqrt{3})$

& $\tan\frac{\pi}{12} = \tan\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{4} - \frac{2\pi/6}{2}\right)$

$\therefore$ do $\sec\frac{\pi}{3} - \tan\frac{\pi}{3} = 2 - \sqrt{3}$

14 a) To prove $\cos 3\theta - \sin 3\theta \equiv (\cos\theta + \sin\theta)(1 - 4\cos\theta\sin\theta)$

<u>LHS</u>: let $c = \cos\theta$, $s = \sin\theta$.

Then $\cos 3\theta = 4c^3 - 3c$ and $\sin 3\theta = -4s^3 + 3s$

$\therefore$ LHS $= 4c^3 - 3c + 4s^3 - 3s$

<u>RHS</u>: $(c+s)(1-4c.s) = c - 4c^2s + s - 4cs^2$

$$= c - 4(1-s^2)s - 4c(1-c^2)$$
$$= c - 4s + 4s^3 - 4c + 4c^3$$

$\therefore$ RHS $= 4c^3 - 3c + 4s^3 - 3s =$ LHS ✓

(b) given $\sec A = \cos B + \sin B$

(i) $\tan^2 A = -1 + \sec^2 A$

$= -1 + (\cos B + \sin B)^2$

$= -1 + \cos^2 B + 2\cos B \sin A + \sin^2 B$

$= -1 + \cos^2 B + \sin^2 B + \sin 2B$

$= \sin 2B$ ✓

(ii)

15 (a) given $\sin x + \sin 2x = \sin 3x$

Rewrite as $\sin x - \sin 3x = -\sin 2x$

Now use factor formula on LHS:

$$\sin x - \sin 3x = 2 \sin \frac{-2x}{2} \cdot \cos \frac{4x}{2}$$

$$= -2 \sin x \cos 2x = -\sin 2x$$

But this does not help so revert to expanding $\sin 2x$ & $\sin 3x$.

Hence $\sin x + 2 \sin x \cos x = -4 \sin^3 x + 3 \sin x$

$\therefore \quad 4 \sin^3 x - 2 \sin x + 2 \sin x \cos x = 0$

$\Rightarrow \quad \sin x \left(4 \sin^2 x + 2\cos x - 2\right) = 0$

$\Rightarrow \quad \sin x \left(4 - 4\cos^2 x + 2\cos x - 2\right) = 0$

$\Rightarrow \quad \sin x \left(-4\cos^2 x + 2\cos x + 2\right) = 0$

$\therefore \quad \sin x = 0$ or $4\cos^2 x - 2\cos x - 2 = 0$

i.e. $2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

Hence when $x = n\pi$; $n = 0, 1, 2, \ldots$; and $\cos x = -\frac{1}{2}$ or $\cos x = 1$

$\Rightarrow \quad x = \pm \frac{2\pi}{3} \pm 2n\pi$ or $x = 2n\pi$

and

$x = \pm \frac{4\pi}{3} \pm 2n\pi$

where $n = 0, 1, 2, \ldots$

(b)

To solve $2\tan x + \sec 2x = 2\tan 2x$;

LHS: $2\tan x + \dfrac{1}{\cos 2x} = 2\tan x + \dfrac{1}{\cos^2 x - \sin^2 x}$

$$= 2\tan x + \frac{1/\cos^2 x}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

$$= 2\tan x + \frac{\sec^2 x}{1-\tan^2 x}$$

$$= 2\tan x + \frac{1+\tan^2 x}{1-\tan^2 x}$$

RHS: $2\tan 2x = 2\left(\dfrac{\tan x + \tan x}{1 - \tan x . \tan x}\right)$

$$= \frac{4\tan x}{1-\tan^2 x}$$

So $\dfrac{4\tan x}{1-\tan^2 x} = 2\tan x + \dfrac{1+\tan^2 x}{1-\tan^2 x}$

$\Rightarrow \quad 4\tan x = 2\tan x(1-\tan^2 x) + 1 + \tan^2 x$

$\Rightarrow \quad 2\tan^3 x - \tan^2 x + 2\tan x - 1 = 0 \quad (*)$

Now either guess by trial and error that $\tan x = \frac{1}{2}$ is a solution to $(*)$, or factorise in terms of $2\tan x - 1$, i.e $(*)$ becomes

$2\tan x . \tan^2 x - \tan^2 x + 2\tan x - 1 = 0$

$\Rightarrow \quad \tan^2 x(2\tan x - 1) + 2\tan x - 1 = 0$

$$\Longrightarrow \quad (2\tan x - 1)(\tan^2 x - 1) = 0$$

$\therefore \quad 2\tan x - 1 = 0 \quad$ OR $\quad \tan^2 x - 1 = 0$

$\Rightarrow \quad \tan x = \frac{1}{2} \quad$ OR $\quad \tan x = \pm 1$

So $\quad x = 26.57° \quad$ OR $\quad x = \pm 45°$

& $\quad x = -153.57° \quad$ & $\quad x = \pm 135°$

But when $x = \pm 45°, \pm 135°$, $\cos 2x = 0$ & $\sec 2x = \infty$!

So the only valid solutions are $x = 26.57°, -153.57°$

(16) given $\sqrt{3}\sin\theta - \cos\theta = R\sin(\theta - \alpha)$

then $\quad \sqrt{3}\sin\theta - \cos\theta = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$\therefore \quad \sqrt{3} = R\cos\alpha \quad$ and $\quad 1 = R\sin\alpha$

$\Rightarrow \quad R = \sqrt{(\sqrt{3})^2 + 1^2} = 2 \quad$ and $\tan\alpha = \frac{1}{\sqrt{3}}$

$$\Rightarrow \alpha = 30°$$

So $\quad \sqrt{3}\sin\theta - \cos\theta = 2\sin(\theta - 30°)$

Now Solve $\quad 4\sin\theta\cos\theta = 2\sin(\theta - 30°)$

i.e $\quad 2\sin 2\theta = 2\sin(\theta - 30°)$

$$\Rightarrow \quad \sin 2\theta = \sin(\theta - 30°)$$

let $\phi = \theta - 30°$

$$\left.\begin{aligned} \therefore \quad 2\theta &= \phi \pm 360n \\ \text{and} \quad 2\theta &= 180 - \phi \pm 360n \end{aligned}\right\} \quad (*)$$

(b)

To Solve $2\tan x + \sec 2x = 2\tan 2x$;

<u>LHS</u>: $2\tan x + \dfrac{1}{\cos 2x} = 2\tan x + \dfrac{1}{\cos^2 x - \sin^2 x}$

$$= 2\tan x + \frac{1/\cos^2 x}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

$$= 2\tan x + \frac{\sec^2 x}{1 - \tan^2 x}$$

$$= 2\tan x + \frac{1+\tan^2 x}{1-\tan^2 x}$$

<u>RHS</u>: $2\tan 2x = 2\left(\dfrac{\tan x + \tan x}{1 - \tan x \cdot \tan x}\right)$

$$= \frac{4\tan x}{1-\tan^2 x}$$

So $\dfrac{4\tan x}{1-\tan^2 x} = 2\tan x + \dfrac{1+\tan^2 x}{1-\tan^2 x}$

$\Rightarrow$ $4\tan x = 2\tan x(1-\tan^2 x) + 1 + \tan^2 x$

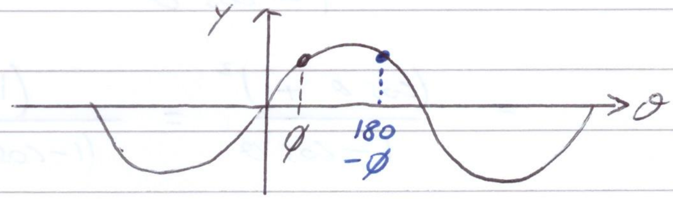
$\Rightarrow$ $2\tan^3 x - \tan^2 x + 2\tan x - 1 = 0$ (*)

Now either guess by trial and error that $\tan x = \frac{1}{2}$ is a solution to (*), or factorise in terms of $2\tan x - 1$, i.e (*) becomes

$$2\tan x \cdot \tan^2 x - \tan^2 x + 2\tan x - 1 = 0$$

$\Rightarrow$ $\tan^2 x(2\tan x - 1) + 2\tan x - 1 = 0$

To see why ⊛ is true look at a sketch of the graph of $\sin\theta$



So the principal values of sin in $0 \le \phi \le 360$ are as seen above

$\therefore \quad 2\theta = \theta - 30^\circ \pm 360^\circ n$

$\Rightarrow \quad \theta = -30 \pm 360^\circ n \quad \Rightarrow \quad \theta = 330^\circ$

and $\quad 2\theta = 180^\circ - (\theta - 30) \pm 360^\circ n$

$\Rightarrow \quad 3\theta = 210^\circ \pm 360^\circ n$

$\therefore \theta = 70^\circ \pm 120^\circ n \quad \Rightarrow \quad \theta = 70^\circ, 190^\circ, 310^\circ$

(17) a) given $(\cot\theta + \operatorname{cosec}\theta)^2 \equiv \dfrac{1+\cos\theta}{1-\cos\theta}$

LHS: $\quad (\cot\theta + \operatorname{cosec}\theta)^2 = \cot^2\theta + 2\cot\theta\operatorname{cosec}\theta + \operatorname{cosec}^2\theta$

But $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, so

$$(\cot\theta + \operatorname{cosec}\theta)^2 = 2\cot^2\theta + 2\cot\theta\operatorname{cosec}\theta + 1$$

$$= 2\frac{\cos^2\theta}{\sin^2\theta} + 2\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta} + 1$$

$$= \frac{2\cos^2\theta + 2\cos\theta + \sin^2\theta}{\sin^2\theta}$$

So $\quad (\cot\theta + \operatorname{cosec}\theta)^2 = \dfrac{\cos^2\theta + 2\cos\theta + 1}{1-\cos^2\theta}$

$$= \frac{(\cos\theta+1)^2}{1-\cos^2\theta} = \frac{(1+\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1+\cos\theta}{1-\cos\theta} \checkmark$$

$$= RHS$$

A quicker way to do This is to Work the RHS as follows:

$$\frac{1+\cos\theta}{1-\cos\theta} = \frac{\operatorname{cosec}\theta + \cot\theta}{\operatorname{cosec}\theta - \cot\theta}$$, by division of top & bottom by $\sin\theta$

Then $\quad \dfrac{1+\cos\theta}{1-\cos\theta} = \dfrac{\operatorname{cosec}\theta+\cot\theta}{\operatorname{cosec}\theta-\cot\theta} \cdot \dfrac{\operatorname{cosec}\theta+\cot\theta}{\operatorname{cosec}\theta+\cot\theta}$

$$= \frac{(\operatorname{cosec}\theta+\cot\theta)^2}{\operatorname{cosec}^2\theta - \cot^2\theta}$$

But since $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ we have $1 = \operatorname{cosec}^2\theta - \cot^2\theta$

Hence $\quad \dfrac{1+\cos\theta}{1-\cos\theta} = (\operatorname{cosec}\theta + \cot\theta)^2$

Now Replace θ by 2θ. Hence $(\cot 2\theta + \operatorname{cosec} 2\theta)^2 = \dfrac{1+\cos 2\theta}{1-\cos 2\theta} \; (*) = \sec 2\theta$

$\therefore \quad \dfrac{\sec 2\theta + 1}{\sec 2\theta - 1} = \sec 2\theta \quad$ by dividing top & bottom of $(*)$ by $\cos 2\theta$.

Hence $\sec 2\theta + 1 = \sec 2\theta (\sec 2\theta - 1)$

$= \sec^2 2\theta - \sec 2\theta$

$\therefore \sec^2 2\theta - 2\sec 2\theta - 1 = 0$

By quadratic formula: $\sec 2\theta = \dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

So $\cos 2\theta = \dfrac{1}{1 \pm \sqrt{2}} = \sqrt{2} - 1, \; -(1+\sqrt{2})$

$\therefore \cos 2\theta = \sqrt{2} - 1$ since cosine cannot be greater (in magnitude) than 1.

So $2\theta = \pm 65.53^\circ \pm 360^\circ n$

$\therefore \theta = \pm 32.76^\circ \pm 180^\circ n$

Hence $\theta = 32.76^\circ, \; 147.23^\circ$

(b) given $\sin 2x + \sin 3x + \sin 5x = 0$ use factor formula to get

$2 \sin \frac{5x}{2} \cos \frac{x}{2} + \sin 5x = 0$

$\therefore 2 \sin \frac{5x}{2} \cos \frac{x}{2} + \sin 2\left(\frac{5x}{2}\right) = 0$

$\Rightarrow 2 \sin \frac{5x}{2} \cdot \cos \frac{x}{2} + 2 \sin \frac{5x}{2} \cos \frac{5x}{2} = 0$

$\therefore 2 \sin \frac{5x}{2} \left(\cos \frac{x}{2} + \cos \frac{5x}{2}\right) = 0$

$\Rightarrow 2 \sin \frac{5x}{2} \left(2 \cos \frac{3x}{2} \cdot \cos x\right) = 0$

Hence $\sin\frac{5x}{2} = 0 \Rightarrow \frac{5x}{2} = \pm n\pi$; $n = 0, 1, 2, \ldots$

$$\Rightarrow x = \pm\frac{2n\pi}{5}$$

or $\cos\frac{3x}{2} = 0 \Rightarrow \frac{3x}{2} = \pm\frac{\pi}{2} \pm n\pi$

$$\Rightarrow x = \pm\frac{\pi}{3} \pm \frac{2n\pi}{3}$$ (Book ANS is incorrect)

& $\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2} \pm n\pi$

(18) given $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$ where

$t \equiv \tan\frac{\theta}{2}$, Then

$$1+\sin\theta = 1+\frac{2t}{1+t^2} = \frac{1+t^2+2t}{1+t^2} = \frac{(t+1)^2}{1+t^2}$$

& $$5+4\cos\theta = 5+4\frac{(1-t^2)}{1+t^2} = \frac{9+t^2}{1+t^2}$$

So $$\frac{1+\sin\theta}{5+4\cos\theta} = \frac{(t+1)^2/(1+t^2)}{(9+t^2)/(1+t^2)} = \frac{(1+t)^2}{9+t^2}$$ ✓

(19) (a) given $\cos^6 x + \sin^6 x \equiv 1 - \frac{3}{4}\sin^2 2x$ to prove/show

<u>LHS</u>: $\cos^6 x + \sin^6 x = (\cos^2 x)^3 + \sin^6 x$

$= (1 - \sin^2 x)^3 + \sin^6 x$

$= (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) + \sin^6 x$

$= 1 - 3\sin^2 x + 3\sin^4 x$

$= 1 - 3\sin^2 x\,(1 - \sin^2 x)$

$= 1 - 3\sin^2 x \cos^2 x$

$= 1 - 3(\sin x \cos x)^2$

$= 1 - \frac{3}{4}(2\sin x \cos x)^2 = 1 - \frac{3}{4}\sin^2 2x$ ✓

(b) given $\sin x + \sin 5x = \sin 3x$

we have $\sin x = \sin 3x - \sin 5x$.

By factor formula we have

$\sin x = 2\cos 4x . \sin(-x) = -2\cos 4x . \sin x$

so $\sin x + 2\cos 4x . \sin x = 0$

$\Rightarrow \sin x(1 + 2\cos 4x) = 0$

$\therefore \sin x = 0$ ~~&~~ / OR $1 + 2\cos 4x = 0 \Rightarrow \cos 4x = -\frac{1}{2}$

Hence in $0^\circ \le x \le 180^\circ$ we have

$\sin x = 0 \quad \Rightarrow \quad x = 0^\circ; \ 180^\circ$

& $\cos 4x = -\frac{1}{2} \quad \Rightarrow \quad 4x = \pm 120^\circ \pm 360^\circ n \quad , \quad n = 0, 1, 2, \ldots$

So $\quad x = \pm 30^\circ \pm 90^\circ n$

Hence $\quad x = 30^\circ, 60^\circ, 120^\circ, 150^\circ$

(c) given $3\cos x + 4\sin x = 2$

let $\quad 3\cos x + 4\sin x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\sin\alpha\cos x,$

hence $\quad 3 = R\sin\alpha \quad$ and $\quad 4 = R\cos\alpha$

$\Rightarrow \quad R = \sqrt{9+16} = 5 \quad$ & $\quad \tan\alpha = \frac{3}{4}$

$\Rightarrow \quad \alpha = 36.87^\circ$

So $\quad 5.\sin(x + 36.87^\circ) = 2$

$\Rightarrow \quad \sin(x + 36.87^\circ) = \frac{2}{5}$

$\therefore \quad x + 36.87^\circ = 23.58^\circ \pm 360^\circ n$

and $\quad x + 36.87^\circ = 156.42^\circ \pm 360 n$

So $\quad x = -13.29^\circ \pm 360^\circ n = -13^\circ 17' \pm 360^\circ n$

or $\quad x = 119.55 \pm 360^\circ n = 119^\circ 33' \pm 360^\circ n$

(20) To Prove $\operatorname{cosec}\theta + \cot\theta \equiv \cot\frac{\theta}{2}$

LHS $\quad \operatorname{cosec}\theta + \cot\theta = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

$$= \frac{\sin\theta + \sin\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{\sin 2(\theta/2)}{1-\cos 2(\theta/2)} \quad (*)$$

$$= \frac{2\sin\theta/2 \cdot \cos\theta/2}{1-\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$= -\frac{\sin\theta/2}{\cos\theta/2} = -\tan\frac{\theta}{2}$$ Not what we want.

From (*) : $\operatorname{cosec}\theta + \cot\theta = \frac{2\sin\theta/2 \cdot \cos\theta/2}{1-(1-2\sin^2\theta/2)}$

$$= \frac{\cos\theta/2}{\sin\theta/2} = \cot\theta/2 \quad ✓$$

Hence (a) $\cot\left(\frac{\pi}{8}\right) = \cot\left(\frac{\pi/4}{2}\right) = \operatorname{cosec}\frac{\pi}{4} + \cot\frac{\pi}{4}$

$= \frac{2}{\sqrt{2}} + 1 = 1 + \sqrt{2}$

& $\cot\left(\frac{\pi}{12}\right) = \cot\left(\frac{\pi/6}{2}\right) = \operatorname{cosec}\frac{\pi}{6} + \cot\frac{\pi}{6}$

$= 2 + \frac{3}{\sqrt{3}} = 2 + \sqrt{3}$

(b) $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta$

$\operatorname{cosec}2\theta = \cot\theta - \cot 2\theta$

$\operatorname{cosec}4\theta = \cot 2\theta - \cot 4\theta$

$\therefore \operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta = \cot\frac{\theta}{2} - \cot\theta$
$+ \cot\theta - \cot 2\theta$
$+ \cot 2\theta - \cot 4\theta$

$= \cot\frac{\theta}{2} - \cot 4\theta.$ (*)

(c) $\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15}$

$= \cot\frac{4\pi}{30} - \cot\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15}$

(here we used $\theta = 4\pi/15$ in (*))

So $\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15}$

$= \cot\frac{4\pi}{30} - \cot\frac{16\pi}{15} + \left(\cot\frac{16\pi}{15} - \cot\frac{32\pi}{15}\right)$

(From $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta$)

So $\operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} + \operatorname{cosec}\frac{32\pi}{15}$

$$= \cot\frac{4\pi}{30} - \cot\frac{32\pi}{15}$$

$$= \cot\frac{4\pi}{30} - \cot\frac{64\pi}{30}$$

$$= \frac{\cos 4\pi/30}{\sin 4\pi/30} - \frac{\cos 64\pi/30}{\sin 64\pi/30}$$

$$= \frac{\sin 64\pi/30 \cdot \cos 4\pi/30 - \sin 4\pi/30 \cdot \cos 64\pi/30}{\sin\frac{4\pi}{30} \cdot \sin\frac{64\pi}{30}}$$

$$= \frac{\sin(64\pi/30 - 4\pi/30)}{\sin\frac{4\pi}{30} \cdot \sin\frac{64\pi}{30}} = \frac{\sin(60\pi/30)}{\sin\frac{4\pi}{30} \cdot \sin\frac{64\pi}{30}}$$

$$= \frac{\sin 2\pi}{\sin\frac{4\pi}{30}\sin\frac{64\pi}{30}}$$

$$= 0 \checkmark$$

(21) To prove $\tan 3\theta = \frac{3t - t^3}{1-3t^2}$ where $t = \tan\theta$:

$$\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \cdot \tan\theta}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta} \cdot \tan\theta}$$

$$= \frac{\frac{2t}{1-t^2} + t}{1 - \frac{2t}{1-t^2} \cdot t}$$

$$\text{So } \tan 3\theta = \frac{2t + t(1-t^2)}{1-t^2 - 2t^2} = \frac{3t - t^3}{1-3t^2} \quad \checkmark$$

(22) given $7\sin x - 24\cos x = R\sin(x-\alpha)$

Then $7\sin x - 24\cos x = R\sin x\cos\alpha - R\sin\alpha\cos x$

So $7 = R\cos\alpha$ and $24 = R\sin\alpha$

$\therefore R = \sqrt{7^2+24^2} = 25$ and $\alpha = \tan^{-1}\dfrac{24}{7} = 73.74^\circ$

So $7\sin x - 24\cos x = 25\sin(x - 73.74^\circ)$

$\therefore \quad 25\sin(x - 73.74^\circ) = 15$

$\Rightarrow \quad \sin(x - 73.74^\circ) = \dfrac{15}{25}$

$\therefore \quad x - 73.74^\circ = 36.87^\circ \pm 360n$

and $x - 73.74^\circ = 143.14^\circ \pm 360n$

So $x = 110.61 = 110^\circ 36'$

& $x = 216.87^\circ = 216^\circ 52'$

for $0 \le x \le 360^\circ$

(b) given $\cos x + \cos y = 1$
and $\sec x + \sec y = 4$,

we have $\cos x + \cos y = 1 \Rightarrow \cos x = 1 - \cos y$ (*)

and $\dfrac{1}{\cos x} + \dfrac{1}{\cos y} = 4 \Rightarrow \dfrac{1}{1-\cos y} + \dfrac{1}{\cos y} = 4$

by (*)

So $$\frac{\cos y + 1 - \cos y}{\cos y(1-\cos y)} = 4$$

$$\Rightarrow \quad \frac{1}{\cos y(1-\cos y)} = 4$$

$$\therefore \quad \frac{1}{4} = \cos y(1-\cos y) = \cos y - \cos^2 y$$

So $$\cos^2 y - \cos y + \frac{1}{4} = 0$$

$$\therefore \quad 4\cos^2 y - 4\cos y + 1 = 0$$

hence $$(2\cos y - 1)(2\cos y - 1) = 0$$

$$\therefore \quad \cos y = \frac{1}{2} \Rightarrow y = 60^\circ$$

And $\cos x = 1 - \cos y$, $\therefore \cos x = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore \quad x = 60^\circ.$$

(23) let $\cos 2x - \sin 2x = R\cos(2x + \alpha)$

$$= R\cos 2x \cos\alpha - R\sin 2x \sin\alpha$$

So $1 = R\cos\alpha$ and $1 = R\sin\alpha$

$\therefore R = \sqrt{1+1} = \sqrt{2}$ and $\alpha = \tan^{-1} 1 = \frac{\pi}{4}$

So $$\cos 2x - \sin 2x = \sqrt{2}\cos\left(2x + \frac{\pi}{4}\right)$$

Then (a) $\sqrt{2}\cos\left(2x + \frac{\pi}{4}\right) = 1$

$$\Rightarrow \cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \quad 2x + \frac{\pi}{4} = \pm\frac{\pi}{4} \pm 2n\pi$$

$$\therefore \quad 2x = \pm 2n\pi$$

or $\quad 2x = -\frac{\pi}{2} \pm 2n\pi$

$$\therefore \quad x = \pm n\pi \quad \text{or} \quad x = -\frac{\pi}{4} \pm n\pi$$

(Book Ans Not complete)

(b) $\sqrt{2}\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}\cos 4x$

$$\therefore \quad \cos\left(2x + \frac{\pi}{4}\right) = \cos 4x$$

So $\quad 2x + \frac{\pi}{4} = \pm 4x \pm 2n\pi$

So $\quad 2x = \pm 4x - \frac{\pi}{4} \pm 2n\pi$

$$\therefore \quad 6x = -\frac{\pi}{4} \pm 2n\pi \quad \text{or} \quad -2x = -\frac{\pi}{4} \pm 2n\pi$$

$$x = -\frac{\pi}{24} \pm \frac{8}{24}n\pi \quad \text{or} \quad x = \frac{\pi}{8} \pm n\pi$$

$$x = \frac{\pi}{24}(\pm 8n - 1) \quad \text{or} \quad x = \frac{\pi}{8}(1 \pm 8\pi)$$

(Book Ans Not complete)

(24)

(a) given $\cos 5x = \cos x$ we have directly

$$5x = \pm x \pm 360^\circ n \quad , \quad n = 0, 1, 2, \dots$$

So $\quad 4x = \pm 360^\circ n$

or $\quad 6x = \pm 360^\circ n$

$\therefore \quad x = \pm 90^\circ n \quad \Rightarrow \quad x = 0^\circ$ and $90^\circ, -90^\circ$

or $\quad x = \pm 60^\circ n \quad \Rightarrow \quad x = 0^\circ$ and $120^\circ, -120^\circ$ and $+60^\circ, -60^\circ$

(b) given $\dfrac{1 + \cos\theta + \sin\theta}{1 - \cos\theta + \sin\theta} \equiv \dfrac{1 + \cos\theta}{\sin\theta}$

let $c = \cos\theta$ and $s = \sin\theta$.

Then $$\frac{1 + c + s}{1 - c + s} = \frac{1 + c + s}{1 - (c - s)} = \frac{1 + (c + s)}{1 - (c - s)} \cdot \frac{1 + (c - s)}{1 + (c - s)}$$

Hence $$\frac{1 + c + s}{1 - c + s} = \frac{1 + (c - s) + (c + s) + c^2 - s^2}{1 - (c - s)^2}$$

$$= \frac{1 + 2c + c^2 + c^2 - s^2}{1 - (c^2 - 2sc + s^2)}$$

$$= \frac{(1 + c)^2 - s^2}{2cs}$$

$$= \frac{(1 + c)^2 - (1 - c^2)}{2cs}$$

$$= \frac{(1 + c)^2 - (1 + c)(1 - c)}{2cs}$$

$$\therefore \quad \frac{1+c+s}{1-c+s} = \frac{(1+c)[1+c-(1-c)]}{2cs}$$

$$= \frac{1+c}{s}$$

$$\therefore \quad \frac{1+\cos\theta+\sin\theta}{1-\cos\theta+\sin\theta} = \frac{1+\cos\theta}{\sin\theta} \checkmark$$

25 a given $4\sin\theta = \sec\theta$

Then $4\cos\theta\sin\theta = 1$

$\therefore \quad 2\sin 2\theta = 1$

$\Rightarrow \quad \sin 2\theta = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \frac{\pi}{6} \pm 2n\pi$

and $\quad 2\theta = \frac{5\pi}{6} \pm 2n\pi$

So $\quad \theta = \frac{\pi}{12} \pm n\pi$

& $\quad \theta = \frac{5\pi}{12} \pm n\pi$

b given $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

Then $\sin\theta + \sin 3\theta + \sin 2\theta + \sin 4\theta = 0$

$\Rightarrow \quad 2\sin 2\theta\cos\theta + 2\sin 3\theta\cos\theta = 0$ by factor formulae

$\Rightarrow \quad 2\cos\theta(\sin 2\theta + \sin 3\theta) = 0$

$\therefore \quad 2\cos\theta \left(2\sin\frac{5\theta}{2} \cdot \cos\theta\right) = 0$

$\Rightarrow \quad 4\cos^2\theta \cdot \sin\frac{5\theta}{2} = 0$

So $\quad \cos^2\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pm n\frac{\pi}{2}$ ✓

OR $\quad \sin\frac{5\theta}{2} = 0 \Rightarrow \frac{5\theta}{2} = \pm n\pi$

$\therefore \theta = \pm \frac{2n\pi}{5}$ ✓

26 (a) given $3\sin 2x = 2\tan x$ we have

$$6\sin x\cos x = 2\frac{\sin x}{\cos x}$$

$\therefore \quad 6\sin x\cos^2 x = 2\sin x$

$\Rightarrow \quad \sin x(3\cos^2 x - 1) = 0$

$\therefore \sin x = 0 \Rightarrow x = 0°, 180°, 360°$

and $3\cos^2 x - 1 = 0 \Rightarrow x = \cos^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$.

For $x = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ we have $x = 54.73°$ & $305.27°$

For $x = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ we have $x = 234.74°, 125.26°$

(b) given $4\cos x - 6\sin x = 5$

Let $\quad 4\cos x - 6\sin x = R\cos(x + \alpha)$

$$= R\cos x\cos\alpha - R\sin x\sin\alpha$$

$\therefore \quad 4 = R\cos\alpha \quad$ and $\quad 6 = R\sin\alpha$

So $\quad R = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$

and $\quad \alpha = \tan^{-1}\frac{6}{4} = 56.31^\circ$

So $\quad 2\sqrt{13}\cos(x+56.31) = 5$

$\therefore \quad \cos(x+56.31^\circ) = \frac{5}{2\sqrt{13}}$

$\Rightarrow \quad x + 56.31^\circ = \pm 46.1^\circ \pm 360^\circ n$

So $\quad x = -10.21 \pm 360^\circ n$

and $\quad x = -102.41 \pm 360^\circ n$

FOR $0 \le x \le 360^\circ$ we have (at $n=1$)

$$x = 349.79^\circ, \ 257.59^\circ$$

27 given $10\sin\left(\frac{\pi}{3}x\right) + 24\cos\left(\frac{\pi}{3}x\right) = 13$

let $10\sin\left(\frac{\pi}{3}x\right) + 24\cos\left(\frac{\pi}{3}x\right) = R\sin\left(\frac{\pi}{3}x + \alpha\right)$

$= R\sin\left(\frac{\pi}{3}x\right)\cos\alpha$

$+ R\sin\alpha\cos\left(\frac{\pi}{3}x\right).$

So $\quad 10 = R\cos\alpha \quad$ and $\quad 24 = R\sin\alpha$

$\Rightarrow \quad R = \sqrt{100+576} = 26$

and $\quad \alpha = \tan^{-1}\left(\frac{24}{10}\right)$

So $\quad 26 \sin\left(\frac{\pi}{3}x + \tan^{-1}\frac{12}{5}\right) = 13$

$\therefore \quad \sin\left(\frac{\pi}{3}x + \tan^{-1}\frac{12}{5}\right) = \frac{1}{2}$

$\Rightarrow \quad \frac{\pi}{3}x + \tan^{-1}\frac{12}{5} = \frac{\pi}{6} \pm 2n\pi$

and $\quad \frac{\pi x}{3} + \tan^{-1}\frac{12}{5} = \frac{5\pi}{6} \pm 2n\pi$

$\therefore \quad x = \frac{1}{2} - \frac{3}{\pi}\tan^{-1}\frac{12}{5} + 6n$

& $\quad x = \frac{5}{2} - \frac{3}{\pi}\tan^{-1}\frac{12}{5} + 6n$

(b) given $\quad 2\cos\theta\cos 2\theta + \sin 2\theta = 2(3\cos^3\theta - \cos\theta)$

Then $\quad 2\cos\theta(2\cos^2\theta - 1) + 2\sin\theta\cos\theta$

$$= 6\cos^3\theta - 2\cos\theta$$

$\therefore \quad 4\cos^3\theta - 2\cos\theta + 2\sin\theta\cos\theta = 6\cos^3\theta - 2\cos\theta$

$\therefore \quad 2\cos^3\theta - 2\sin\theta\cos\theta = 0$

$\Rightarrow \quad \cos\theta(\cos^2\theta - \sin\theta) = 0$

$\Rightarrow \cos\theta = 0 \quad$ or $\quad \cos^2\theta - \sin\theta = 0$

i.e $\quad 1 - \sin^2\theta - \sin\theta = 0$

$\therefore \quad \sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\sin\theta \neq -\frac{1}{2} - \frac{\sqrt{5}}{2}$ because This less Than -1

So $\sin\theta = -\frac{1}{2} + \frac{\sqrt{5}}{2}$

$\therefore$ we have $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

& $\sin\theta = -\frac{1}{2} + \frac{\sqrt{5}}{2} \Rightarrow \theta = 0.67^c, 2.48^c.$

28 (a) given $2\sin\theta = \sqrt{3}\cdot\tan\theta$

$$\therefore \quad 4\sin^2\theta = 3\tan^2\theta$$

$$\Rightarrow 4(1-\cos^2\theta) = 3\frac{\sin^2\theta}{\cos^2\theta}$$

So $$4\cos^2\theta - 4\cos^4\theta = 3\sin^2\theta$$
$$= 3(1-\cos^2\theta)$$
$$= 3 - 3\cos^2\theta$$

$$\therefore \quad 4\cos^4\theta - 7\cos^2\theta + 3 = 0$$

$$\Rightarrow (4\cos^2\theta - 3)(\cos^2\theta - 1) = 0$$

$$\therefore \quad \cos^2\theta = \frac{3}{4} \quad \&/\text{or} \quad \cos^2\theta = 1$$

$$\Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2} \quad \&/\text{or} \quad \cos\theta = \pm 1$$

So $\theta = \pm\frac{\pi}{6} \pm 2n\pi$ or $\theta = \pm n\pi$

(b) given $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

Then $4\sin\theta - 3\cos\theta = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

So $4 = R\cos\alpha$ and $3 = R\sin\alpha$

$\therefore R = \sqrt{4^2+3^2} = 5$ and $\alpha = \tan^{-1}\frac{3}{4} = 36.87^\circ$

$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.87^\circ)$

i) So $5\sin(\theta - 36.87^\circ) = 3$

$\Rightarrow \sin(\theta - 36.87^\circ) = \frac{3}{5}$

$\Rightarrow \theta - 36.87^\circ = 36.87^\circ \pm 360n \quad , n = 0, 1, 2, \ldots$

and $\theta - 36.87^\circ = 143.13^\circ \pm 360n \quad , n = 0, 1, 2, \ldots$

For $0^\circ < \theta < 360^\circ$ we have $\theta = 73.74^\circ, 180^\circ$

ii) $$\frac{1}{4\sin\theta - 3\cos\theta + 6} = \frac{1}{5\sin(\theta - 36.87^\circ) + 6}$$

Max value occurs when denominator is minimum.
This min occurs when $\sin = -1$

So max value $= \frac{1}{-5+6} = 1$

min value occurs when denominator is maximum. This maximum occurs when $\sin = +1$

So min value $= \frac{1}{5+6} = \frac{1}{11}$

(29) (a) given $x = 2\sin\left(nt + \frac{\pi}{3}\right)$ & $y = 4\sin\left(nt + \pi/6\right)$

we have $x = 2\sin nt \cos\frac{\pi}{3} + 2\sin\frac{\pi}{3}\cos nt$

$= \sin nt + \sqrt{3}\cos nt$ (a)

& $y = 4\sin nt\cos\frac{\pi}{6} + 4\sin\frac{\pi}{6}\cos nt$

$= 2\sqrt{3}\sin nt + 2\cos nt$ (b)

By (a) : $\sin nt = x - \sqrt{3}\cos nt$
(b) : $\sin nt = \frac{y}{2\sqrt{3}} - \frac{1}{\sqrt{3}}\cos nt$

So $x - \sqrt{3}\cos nt = \frac{y}{2\sqrt{3}} - \frac{1}{\sqrt{3}}\cos nt$

$\Rightarrow x - \frac{y}{2\sqrt{3}} = \cos nt\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\cos nt$

$\Rightarrow \frac{\sqrt{3}}{2}x - \frac{y}{4} = \cos nt$ (c)

Also, by (a) : $\cos nt = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\sin nt$

by (b) : $\cos nt = \frac{1}{2}y - \sqrt{3}\sin nt$

So $\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\sin nt = \frac{1}{2}y - \sqrt{3}\sin nt$

$\Rightarrow \frac{1}{\sqrt{3}}x - \frac{1}{2}y = \sin nt\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -\frac{2}{\sqrt{3}}\sin nt$

$\therefore -\frac{1}{2}x + \frac{\sqrt{3}}{4}y = \sin nt$ (d)

Now Square (c) & (d) And add:

$$\left(\frac{\sqrt{3}}{2}x - \frac{y}{4}\right)^2 + \left(-\frac{1}{2}x + \frac{\sqrt{3}}{4}y\right)^2 = \cos^2 wt + \sin^2 wt = 1$$

Expand & Simplify to get

$$4x^2 + y^2 - 2\sqrt{3}\,xy = 4.$$

(b) given $3\cos^2\theta + 5\sin\theta - 1 = 0$

we have $3(1-\sin^2\theta) + 5\sin\theta - 1 = 0$

So $3\sin^2\theta - 5\sin\theta - 2 = 0$

$\Rightarrow (3\sin\theta + 1)(\sin\theta - 2) = 0$

$\therefore 3\sin\theta + 1 = 0$ or $\sin\theta - 2 = 0$

$\Rightarrow \sin\theta = -\frac{1}{3}$ or $\sin\theta = 2 \rightarrow$ Not valid so ignore

$\therefore \theta = -19.47°, -160.53°$

For $0 < \theta < 360°$

$\theta = 19.47°, 160.53°$

(30) To Show That $\sin^2 2\theta\,(\cot^2\theta - \tan^2\theta) = 4\cos 2\theta$

(a)

From LHS we have

$$(2\sin\theta\cos\theta)^2(\cot^2\theta - \tan^2\theta) = 4\sin^2\theta\cos^2\theta(\cot^2\theta - \tan^2\theta)$$

So
$$\sin^2 2\theta(\cot^2\theta - \tan^2\theta) = 4\sin^2\theta\cos^2\theta\cot^2\theta - 4\sin^2\theta\cos^2\theta\tan^2\theta$$

$$= 4\cos^4\theta - 4\sin^2\theta$$

$$= 4(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= 4(\cos^2\theta - \sin^2\theta) = 4\cos 2\theta \checkmark$$

(b) given $\sec\theta\tan\theta = 2$ Then

$$\frac{\sin\theta}{\cos^2\theta} = 2 \Rightarrow \sin\theta = 2\cos^2\theta$$
$$= 2(1-\sin^2\theta)$$
$$= 2 - 2\sin^2\theta$$

$$\therefore \quad 2\sin^2\theta + \sin\theta - 2 = 0$$

So $$\sin\theta = \frac{-1 \pm \sqrt{1+16}}{4} = -\frac{1}{4} \pm \frac{\sqrt{17}}{4}$$

But $\sin\theta = \dfrac{-1-\sqrt{17}}{4} < -1$, so is not valid

$$\therefore \theta = \sin^{-1}\left(\frac{-1+\sqrt{17}}{4}\right) = 51.33°,\ 128.67°$$

(31) $\cos(A+B) = \cos A.\cos B - \sin A.\sin B$ (i)

$\cos(A-B) = \cos A \cos B + \sin A.\sin B$ (ii)

(a) given $\cos x \cos y = 0.6$ & $\sin x \sin y = 0.2$
we have by (i) & (ii)

$$\frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right] = \cos x \cos y = 0.6$$

& $$\frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] = \sin x . \sin y = 0.2$$

$\therefore$ $\cos(x+y) + \cos(x-y) = 1.2$ (iii)

& $\cos(x-y) - \cos(x+y) = 0.4$ (iv)

adding (iii) & (iv) : $2\cos(x-y) = 1.6 \Rightarrow \cos(x-y) = 0.8$

Subtracting (iv) FROM (iii) : $2\cos(x+y) = 0.8 \Rightarrow \cos(x+y) = 0.4$

hence, in $0 < x, y < 90°$,

$$x + y = 66.42°$$
& $$x - y = 36.87°$$

adding : $2x = 103.29° \Rightarrow x = 51.65°$

Subtracting : $2y = 29.55° \Rightarrow y = 14.78°$

(b) $\cos 3x = \cos(2x+x) = \cos 2x \cos x - \sin 2x \sin x$

$$= (2\cos^2 x - 1)\cos x - 2\sin x \cos x . \sin x$$

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 4\cos^3 x - 3\cos x \quad \checkmark$$

So $\quad 2\cos 3x + \cos 2x + 1 = 0$

$$\Rightarrow \quad 4\cos^3 x - 3\cos x + (2\cos^2 x - 1) + 1 = 0$$

$$\therefore \quad \cos x\,(4\cos^2 x + 2\cos x - 3) = 0$$

From the quadratic we have

$$\cos x = \frac{-2 \pm \sqrt{4+48}}{8} = -\frac{1}{4} \pm \frac{\sqrt{13}}{4}$$

So $\quad \cos x = 0 \quad$ or $\quad \cos x = -\frac{1}{4} + \frac{\sqrt{13}}{4} \quad$ (since $-\frac{1}{4} - \frac{\sqrt{13}}{4} < -1$

$\Rightarrow \quad x = \pm 90^\circ \quad$ or $\quad x = \pm 49.35^\circ \quad$ & $\cos x$ cannot be less than -1)

Book Ans is incorrect.

(32) (a) i) given $\cos 2x + \sin x = 0$

Then $\quad 1 - 2\sin^2 x + \sin x = 0$

$$\Rightarrow \quad 2\sin^2 x - \sin x - 1 = 0$$

$$\therefore \quad (2\sin x + 1)(\sin x - 1) = 0$$

$$\Rightarrow \quad 2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$

$$\therefore \quad \sin x = -\tfrac{1}{2} \quad \text{or} \quad \sin x = 1$$

$$\Rightarrow \quad x = -30^\circ \quad \text{or} \quad x = 90^\circ$$

For $\quad 0^\circ < x < 360^\circ : \quad x = 210^\circ, 330^\circ, 90^\circ$

ii) given $\sin x - \sin 2x + \sin 3x = 0$

Then $\sin x + \sin 3x - \sin 2x = 0$

$\Rightarrow \quad 2\sin 2x \cos x - \sin 2x = 0$ by factor formula

$\therefore \quad \sin 2x\,(2\cos x - 1) = 0$

$\Rightarrow \quad \sin 2x = 0$ or $2\cos x - 1 = 0$

$\therefore \quad 2x = 0;\ 180^\circ;\ 360^\circ,\ 540^\circ \ ; \ \cos x = \frac{1}{2}$

So in $(0^\circ, 360^\circ)$: $x = 90^\circ,\ 180^\circ,\ 270^\circ \ ; \ x = 60^\circ,\ 300^\circ$

(b) It is not obvious how you would start here, unless you had seen a previous example. So this is how you can solve this problem:

$A = 36^\circ$, hence $5A = 180^\circ$

So $5A = 2A + 3A = 180^\circ \Rightarrow 2A = 180^\circ - 3A$

So $\sin 2A = \sin(180 - 3A) = \sin 180^\circ \cos 3A - \sin 3A \cos 180^\circ$

$= 0 - (-\sin 3A)$

$= \sin 3A$ ✓

Continuing: $2\sin A \cos A = 3\sin A - 4\sin^3 A$

So $2\sin A \cos A - 3\sin A + 4\sin^3 A = 0$

$\Rightarrow \quad \sin A\,(2\cos A - 3 + 4\sin^2 A) = 0$

$\therefore \quad \sin A\left(2\cos A - 3 + 4(1-\cos^2 A)\right) = 0$

$\Rightarrow \quad \sin A = 0$ or $4\cos^2 A - 2\cos A - 1 = 0$

But $A = 36^\circ \Rightarrow \sin 36^\circ \neq 0$ so $4\cos^2 A - 2\cos A - 1 = 0$ only

$$\therefore \quad \cos A = \frac{2 \pm \sqrt{4+16}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

But $\dfrac{1-\sqrt{5}}{4}$ is negative & $\therefore$ does not give an angle for cosine in quadrant I (since 36° is in quadrant I)

$$\therefore \quad \cos A = \frac{1+\sqrt{5}}{4} \checkmark$$

(c) Start with $\tan(A+B)$; so

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Since given $\tan A = \dfrac{n}{n+1}$ we have

$$\tan(A+B) = \frac{n/(n+1) + \tan B}{1 - \frac{n}{n+1}\tan B} = \tan\frac{\pi}{4} = 1$$

$$\therefore \quad \frac{n}{n+1} + \tan B = 1 - \frac{n}{n+1}\tan B$$

$$\Rightarrow \quad \tan B\left(1 + \frac{n}{n+1}\right) = 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

So $\quad \tan B = \left(\dfrac{1}{n+1}\right) \div \left(\dfrac{2n+1}{n+1}\right) = \dfrac{1}{2n+1}$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{n}{n+1} - \frac{1}{2n+1}}{1 + \left(\frac{n}{n+1}\right)\left(\frac{1}{2n+1}\right)}$$

$$= \frac{(2n+1)n - (n+1)}{(n+1)(2n+1) + n} = \frac{2n^2+1}{2n^2+4n+1}$$

(33) when θ is small we have

$$\sin\theta \simeq \theta \ , \quad \cos\theta \simeq 1 - \frac{\theta^2}{2} \ , \quad \tan\theta \simeq \theta$$

$$\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

(a) $\sin\theta + \cos\theta \simeq 1 + \theta - \frac{\theta^2}{2}$

(b) $$\frac{2\tan\theta - \theta}{\sin 2\theta} \simeq \frac{2\theta - \theta}{2\sin\theta\cos\theta} = \frac{\theta}{2\theta(1-\theta^2/2)}$$

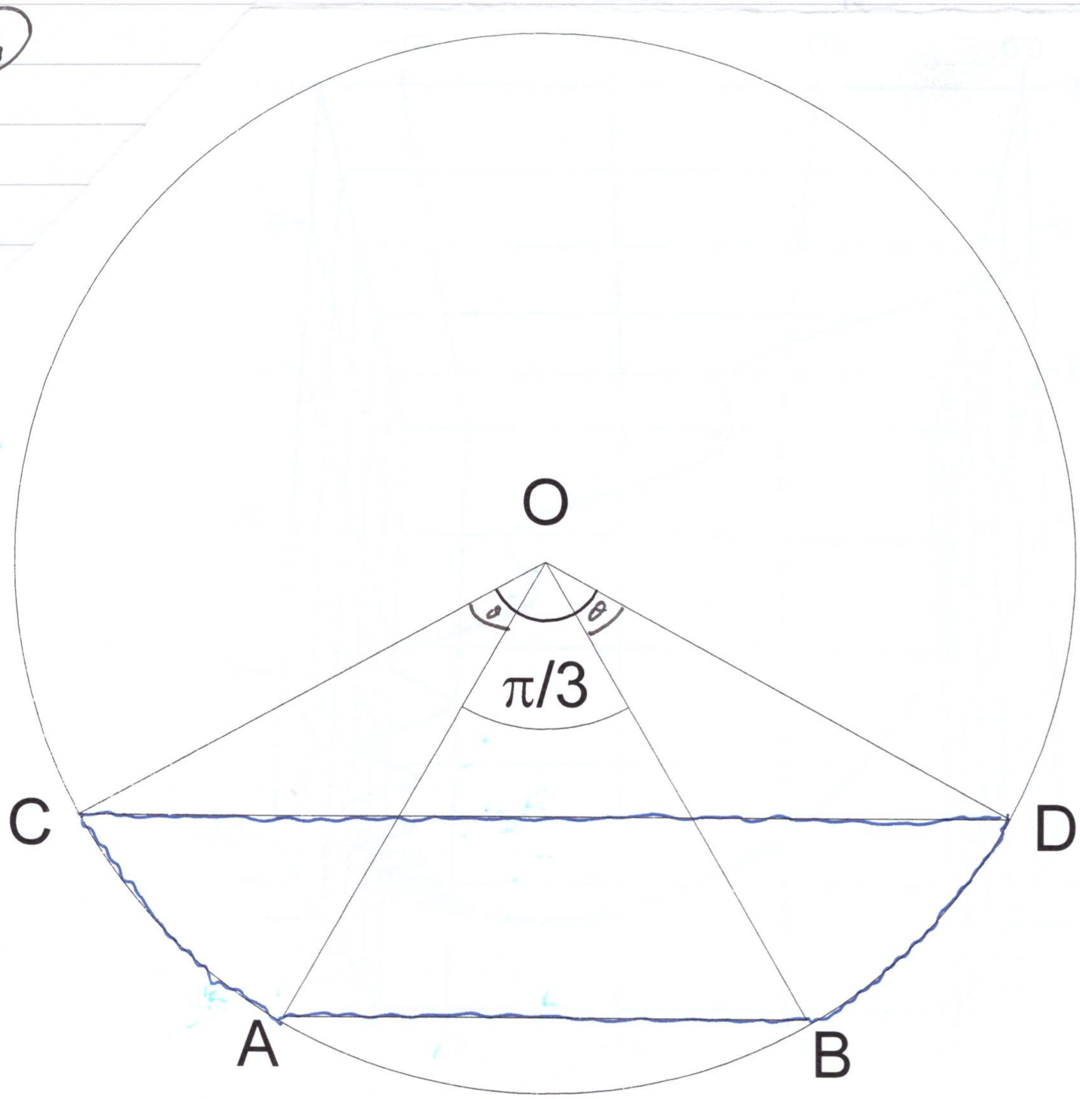
(c) $$\cot\theta\,(1-\cos\theta) = \frac{\cos\theta}{\sin\theta}(1-\cos\theta)$$

$$\simeq \frac{1-\theta^2/2}{\theta}\left(1-(1-\theta^2/2)\right)$$

$$= \frac{\theta^2/2 \cdot (1-\theta^2/2)}{\theta} = \frac{\theta}{2}\left(1-\frac{\theta^2}{2}\right)$$

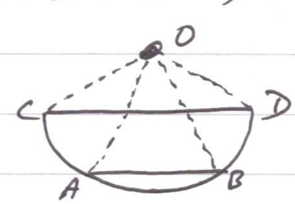
(d) $\dfrac{\sqrt{2} - \sin\theta}{\cos\theta} \simeq \dfrac{\sqrt{2} - \theta}{1 - \theta^2/2} = \dfrac{2\sqrt{2} - 2\theta}{1 - \theta^2}$

(34)



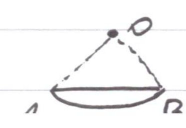
The area we are looking for is the area outlined in Blue.
Of the many ways we can look at this diagram, the way we want is

① Sector COD − triangle COD →



and

② Sector AOB − triangle AOB →



Area of (Sector COD − triangle COD) $= \frac{1}{2}R^2\left(\frac{\pi}{3}+2\theta\right) - \frac{1}{2}R^2\sin\left(\frac{\pi}{3}+2\theta\right)$

Area of (Sector AOB − triangle AOB) $= \frac{1}{2}R^2\left(\frac{\pi}{3}\right) - \frac{1}{2}R^2\sin\frac{\pi}{3}$

$$= \frac{1}{2}R^2 \cdot \frac{\pi}{3} - \frac{\sqrt{3}}{4}R^2$$

So Required area $=$ (1) $-$ (2)

$$= \frac{1}{2}R^2\left(\frac{\pi}{3}+2\theta\right) - \frac{1}{2}R^2\sin\left(\frac{\pi}{3}+2\theta\right)$$

$$- \frac{1}{2}R^2 \cdot \frac{\pi}{3} + \frac{\sqrt{3}}{4}R^2$$

$$= \frac{1}{4}R^2\left[2\left(\frac{\pi}{3}+2\theta\right) - 2\sin\left(\frac{\pi}{3}+2\theta\right) - 2\frac{\pi}{3} + \sqrt{3}\right]$$

$$= \frac{1}{4}R^2\left(4\theta + \sqrt{3} - 2\sin\left(\frac{\pi}{3}+2\theta\right)\right) \quad \circledast$$

Now, $\sin\left(\frac{\pi}{3}+2\theta\right) = \sin\frac{\pi}{3} \cdot \cos 2\theta + \sin 2\theta \cos\frac{\pi}{3}$

$$= \frac{\sqrt{3}}{2} \cdot \cos 2\theta + \frac{1}{2}\sin 2\theta$$

$$= \frac{\sqrt{3}}{2}\left(1 - 2\sin^2\theta\right) + \sin\theta \cdot \cos\theta$$

$$\simeq \frac{\sqrt{3}}{2}\left(1 - 2\theta^2\right) + \theta\left(1 - \frac{\theta^2}{2}\right)$$

$$\simeq \frac{\sqrt{3}}{2} - \sqrt{3}\theta^2 + \theta - \frac{\theta^3}{2}$$

So By $\circledast$, Area $= \frac{1}{4}R^2\left[4\theta + \sqrt{3} - 2\left(\frac{\sqrt{3}}{2} - \sqrt{3}\theta^2 + \theta - \frac{\theta^2}{2}\right)\right]$

Any term in θ^3 can be ignored, since θ is small. Hence

$$\text{Area} \simeq \frac{1}{4}R^2\left(2\theta + 2\sqrt{3}\theta^2\right) = \frac{1}{2}R^2\theta + \frac{\sqrt{3}}{2}R^2\theta^2$$

$$\Rightarrow a = 0, \quad b = \frac{1}{2}R^2, \quad c = \frac{\sqrt{3}}{2}R^2$$

(35) given $(2-\tan\theta)(1+\sin 2\theta)-2=0$

& $\tan\theta \equiv t$, $\sin 2\theta \equiv \frac{2t}{1+t^2}$

we have

$$(2-t)\left(1+\frac{2t}{1+t^2}\right)-2=0$$

$\therefore \quad (2-t)(1+t^2+2t)-2(1+t^2)=0$

$\Rightarrow \quad 2+2t^2+4t-t-t^3-2t^2-2-2t^2=0$

$\therefore \quad t^3+2t^2-3t=0$

$\Rightarrow \quad t(t^2+2t-3)=0$

$\therefore \quad t\,(t+3)(t-1)=0$

hence $\quad t=\tan\theta=0 \quad \Rightarrow \quad \theta=0$

& $\quad t\ 1=0 \Rightarrow \tan\theta-1=0 \Rightarrow \tan\theta=1$

$\Rightarrow \theta=\frac{\pi}{4}$

& $\quad t+3=0 \Rightarrow \tan\theta+3=0 \Rightarrow \tan\theta=-3$ Not valid

When θ is small : $(2-\tan\theta)(1+\sin 2\theta)-2=(2-\tan\theta)(1+2\sin\theta\cos\theta)-2=0$

$\therefore \quad (2-\tan\theta)(1+\sin 2\theta)-2 \simeq (2-\theta)\left(1+2\theta\left(1-\frac{\theta^2}{2}\right)\right)-2$

$=(2-\theta)(1+2\theta-\theta^3)-2$

$=2+4\theta-2\theta^3-\theta+2\theta^2+\theta^4-2$

$$\therefore \quad (2-\tan\theta)(1+\sin 2\theta)-2 \simeq 3\theta+2\theta^2-2\theta^3+\theta^4$$

But since θ is small

$$(2-\tan\theta)(1+\sin 2\theta)-2 \simeq 3\theta \quad \checkmark$$

36 To Prove $\tan^{-1}x+\tan^{-1}y \equiv \tan^{-1}\dfrac{x+y}{1-xy}$

Let $A=\tan^{-1}x$ & $B=\tan^{-1}y$

So $x=\tan A$ & $y=\tan B$ $\circledast$

Now, $\tan\left(\tan^{-1}x+\tan^{-1}y\right) \equiv \tan\left[\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)\right]$

i.e $\tan(A+B) \equiv \dfrac{x+y}{1-xy}$

So $\dfrac{\tan A+\tan B}{1-\tan A\tan B} \equiv \dfrac{x+y}{1-xy}$

By $\circledast$: $\dfrac{x+y}{1-xy} \equiv \dfrac{x+y}{1-xy}$ $\checkmark$

Now let $C=\tan^{-1}z \Rightarrow z=\tan C$

So $\tan\left(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z\right)=\tan\frac{\pi}{2}$

$\Rightarrow \tan(A+B+C)=\tan\frac{\pi}{2}$

$\Rightarrow \dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}=\tan\frac{\pi}{2}$

So
$$\frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \cdot \tan C} = \tan \frac{\pi}{2}$$

i.e
$$\frac{\dfrac{x+y}{1-xy} + z}{1 - \dfrac{x+y}{1-xy} \cdot z} = \tan \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{x+y+z(1-xy)}{(1-xy)-(x+y)z} = \tan \frac{\pi}{2}$$

Now, $\tan \frac{\pi}{2}$ = "∞" which means That denominator $= 0$:

i.e. $1 - xy - xz - yz = 0$

$\Rightarrow \quad xy + xz + yz = 1$ ✓

(37) Let $A = \tan^{-1} \frac{1}{2}$ & $B = \tan^{-1} \frac{1}{3}$ & $C = \sin^{-1} x$

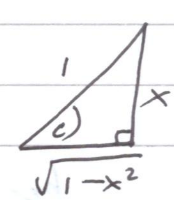
So $\tan A = \frac{1}{2}$ & $\tan B = \frac{1}{3}$ & $\sin C = x$

Then $\tan\left(\tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{3}\right) = \tan(\sin^{-1} x)$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow \quad \tan(A - B) = \tan C$

LHS: $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A . \tan B} = \dfrac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2}.\frac{1}{3}} = \dfrac{1/6}{7/6} = \dfrac{1}{7}$

RHS: $\sin C = x$:



hence $\tan C = \dfrac{x}{\sqrt{1-x^2}}$

$$\therefore \quad \frac{x}{\sqrt{1-x^2}} = \frac{1}{7}$$

$$\therefore \quad \sqrt{1-x^2} = 7x \implies 1-x^2 = 49x^2$$

$$\therefore \quad 1 = 50x^2$$

$$\implies \frac{2}{100} = x^2 \implies x = \pm\frac{1}{10}\sqrt{2}$$

Substituting Each x value into The original Equation Shows That the only valid value is

$$x = +\frac{1}{10}\sqrt{2}.$$

(38) given $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

Then $2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$

let $\alpha = \tan^{-1}\left(\frac{1-x}{1+x}\right)$ & $\beta = \tan^{-1}x$

$\therefore \quad \tan\alpha = \frac{1-x}{1+x}$ & $\tan\beta = x$ ⊛

So $\tan\left[2.\tan^{-1}\left(\frac{1-x}{1+x}\right)\right] = \tan\left(\tan^{-1}x\right)$

$\implies \tan(2\alpha) = \tan\beta$

$$\therefore \quad \frac{2\tan\alpha}{1-\tan^2\alpha} = \tan\beta$$

$$\implies \frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = x \quad \text{by ⊛}$$

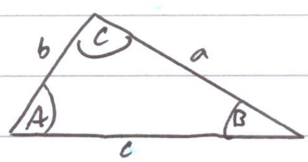
$$\therefore \quad \frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}} = x$$

So $$\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2} = x \Rightarrow \frac{2(1-x^2)}{4x} = x$$

So $$1-x^2 = 2x^2 \Rightarrow 3x^2 = 1$$

$$\therefore \quad x = \pm \frac{1}{\sqrt{3}}$$

(39) Consider The following triangle



By The Sine Rule we have $\frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\therefore \quad b = k.\sin B$

& $c = k.\sin C$

$$\therefore \quad \frac{b-c}{b+c} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)} = \frac{2\sin\left(\frac{B-C}{2}\right).\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right).\cos\left(\frac{B-C}{2}\right)},$$

by the factor formula. So

$$\frac{b-c}{b+c} = \tan\left(\frac{B-C}{2}\right).\cot\left(\frac{B+C}{2}\right)$$

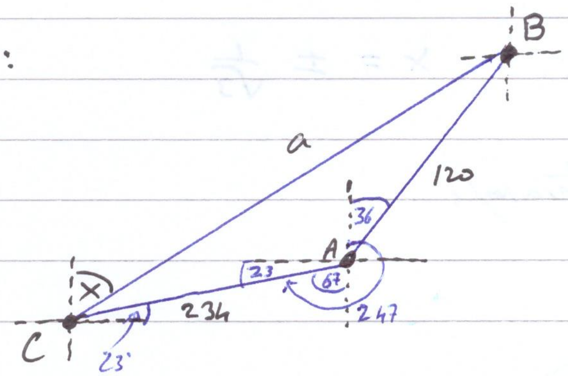
But in a plane triangle, $A+B+C=\pi$

So $\dfrac{B+C}{2} = \dfrac{\pi}{2} - \dfrac{A}{2}$

So $\dfrac{b-c}{b+c} = \tan\left(\dfrac{B-C}{2}\right) \cdot \cot\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right)$

$= \tan\left(\dfrac{B-C}{2}\right) \cdot \tan\left(\dfrac{A}{2}\right)$ ✓

<u>Now</u> :



<u>Find x & a</u>

$\angle CAB = 23 + 90 + 36 = 149°$

So $\tan \dfrac{A}{2} \cdot \tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}$

$\Rightarrow$

$\tan\left(\dfrac{149}{2}\right) \cdot \tan\left(\dfrac{B-C}{2}\right) = \dfrac{234-120}{234+120} = 0.322$

$\therefore \tan \dfrac{B-C}{2} = 0.089 \Rightarrow \dfrac{B-C}{2} = 5.103°$

$\therefore B - C = 10.207°$

Also $A + B + C = 180° \Rightarrow B + C = 180° - A = 180 - 149 = 31°$

$\therefore$ solve $B - C = 10.207$ & $B + C = 31°$ to get

$B = 20.535°$ & $C = 10.604°$

So to find x do : $90° = x + C + 23°$

$$= x + 10.604 + 23$$

$\Rightarrow$ $x = 56.397°$; This is the bearing of B from C.

To find the distance BC use the tan identity above, recast so that we use known info, ie use

$$\tan\left(\frac{B}{2}\right) \cdot \tan\left(\frac{A-C}{2}\right) = \frac{a-c}{a+c}$$

$$\therefore \quad \tan\left(\frac{20.535}{2}\right) \cdot \tan\left(\frac{149-10.604}{2}\right) = \frac{a-120}{a+120}$$

$$\therefore \quad 0.4768 = \frac{a-120}{a+120}$$

$$\Rightarrow \quad a = \frac{-120 - 120(0.4768)}{0.4768 - 1} = 338.73\,\text{km}$$ From B to C.

Note : we could also have used the Cosine Rule to find length a & then the Sine Rule to find angle C, whence angle x.